① TDC part III paper V MATHEMATICAL PHYSICS Group A

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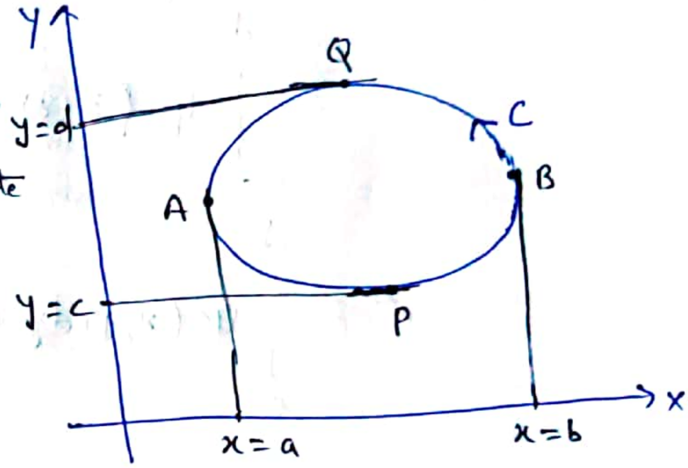
Green' Theorem :- Green theorem is a relation between line integral and surface integral.

Statement :- If $\phi(x,y)$, $\psi(x,y)$, $\frac{\delta\phi}{\delta y}$ and $\frac{\delta\psi}{\delta x}$ are continuous functions over a region R bounded by a simple closed curve c in x-y plane then

$$\oint_C (\phi\, dx + \psi\, dy) = \iint_R \left(\frac{\delta\psi}{\delta x} - \frac{\delta\phi}{\delta y}\right) dx\, dy .$$

where C is traversed in the +ve (anticlockwise) direction.

proof :- Suppose that the region R is such that any line parallel to either coordinate axis meets the boundary C in atmost two points. Let the region included between the lines $x=a$, $x=b$, $y=c$ and $y=d$ are as shown in figure.



Let the equation of the curves C_1 (APB) and C_2 (BQA) be $y = f_1(x)$ and $y = f_2(x)$ respectively.

Now we have

$$\iint_R \frac{\delta\phi}{\delta y}\, dx dy = \int_{x=a}^{x=b} \left(\int_{y=f_1(x)}^{y=f_2(x)} \frac{\delta\phi}{\delta y}\, dy \right) \cdot dx = \int_a^b \Big[\phi(x,y) \Big]_{y=f_1(x)}^{y=f_2(x)} dx$$

$$= \int_a^b \Big[\phi\{x, f_2(x)\} - \phi\{x, f_1(x)\} \Big] dx$$

$$= \int_a^b \phi(x,y)\, dx - \int_a^b \phi(x,y)\, dx$$

$$= - \int_b^a \phi(x,y)\, dx - \int_a^b \phi(x,y)\, dx$$

$$\Rightarrow \iint_R \frac{\delta\phi}{\delta y} dx dy = -\left[\int_{C_2} \phi(x,y)\,dx + \int_{C_1} \phi(x,y)\,dx\right] = -\oint_c \phi(x,y)\,dx$$

$$\Rightarrow -\iint_R \frac{\delta\phi}{\delta y} dx dy = +\oint_c \phi(x,y)\,dx \quad \text{------} \quad (1)$$

Again, let the equations of curve C_1' (QAP) and C_2' (PBQ) be $x = g_1(y)$ and $x = g_2(y)$ respectively, then we have

$$\iint_R \frac{\delta\psi}{\delta x} dx dy = \int_{y=c}^{y=d} \left(\int_{x=g_1(y)}^{x=g_2(y)} \frac{\delta\psi}{\delta x} dx \right) dy = \int_c^d \Big[\psi(x,y)\Big]_{x=g_1(y)}^{x=g_2(y)} \cdot dy$$

$$= \int_c^d \psi\{g_2(y), y\}\,dy - \int_c^d \psi\{g_1(y), y\}\,dy$$

$$= \int_c^d \psi(x,y)\,dy + \int_d^c \psi(x,y)\,dy$$

$$= \int_{C_2'} \psi(x,y)\,dy + \int_{C_1'} \psi(x,y)\,dy$$

$$\iint_R \frac{\delta\psi}{\delta x} dx dy = \oint_C \psi(x,y)\,dy \quad \text{------} \quad (2)$$

$$(1) + (II) \Rightarrow -\iint_R \frac{\delta\phi}{\delta y} dx dy + \iint_R \frac{\delta\psi}{\delta x} dx dy = \oint_c \phi(x,y)\,dx + \oint_c \psi(x,y)\,dy$$

$$\Rightarrow \oint(\phi\,dx + \psi\,dy) = \iint_R \left(\frac{\delta\psi}{\delta x} - \frac{\delta\phi}{\delta y}\right) dx.dy$$ proved.